Injective Envelopes and Local Multiplier Algebras of Some Spatial Continuous Trace C*-algebras*

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Abstract

A precise description of the injective envelope of a spatial continuous trace C*-algebra A over a Stonean space Δ is given. The description is based on the notion of a weakly continuous Hilbert bundle, which we show herein to be a Kaplansky–Hilbert module over the abelian AW*-algebra $C(\Delta)$. We then use the description of the injective envelope of A to study the first- and second-order local multiplier algebras of A. In particular, we show that the second-order local multiplier algebra of A is precisely the injective envelope of A.

Introduction

A commonly used technique in the theory of operators algebras is to study a given C*-algebra A by one or more of its enveloping algebras. Well known examples of such enveloping algebras are the enveloping von Neumann algebra A^{**} and the multiplier algebra $M(A)$. In this paper we consider two others: the local multiplier algebra $M_{\mathrm{loc}}(A)$ and the injective envelope $I(A)$, both

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of which have received considerable study and application in recent years (see, for example, [1, 6, 7, 9, 11, 19, 21, 22]).

The C*-algebras $M_{\mathrm{loc}}(A)$ and $I(A)$ are difficult to determine precisely, even for fairly rudimentary types of C*-algebras A. For instance, if we denote by $C_0(T)$ an abelian C*-algebra and by $K(H)$ the ideal of compact operators over H, their local multiplier algebra and injective envelope have been readily computed; but the injective envelope of $C_0(T)\otimes K(H)$ is much more difficult to describe: see [15] for an abstract description and [3, 4] for a somewhat more concrete one.

Our first goal in the present paper is to make a further contribution to the issue of the determination of $I(A)$ and $M_{\mathrm{loc}}(A)$ from A by considering continuous trace C*-algebras studied by Fell [10] that arise from continuous Hilbert bundles. The class of such algebras contains in particular all C*-algebras of the form $C_0(T) \otimes K(H)$, which were studied in [4]. Because the centres of $I(A)$ and $M_{\mathrm{loc}}(A)$ are AW*-algebras, and thus have Stonean maximal ideal spaces, we restrict ourselves in this paper to locally compact Hausdorff spaces T that are Stonean. In so doing, we establish an important first step toward a complete analysis, in the case of non-Stonean T, of the C*-algebras $I(A)$, $M_{\mathrm{loc}}(A)$, and $M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(A)\right)$ for spatial continuous trace C*-algebras A with spectrum T. As the passage from general T to Stonean T involves a number of technicalities, the application of the main results herein to the case of arbitrary locally compact Hausdorff spaces T will be deferred to a subsequent article.

Our second goal is to study and use the notion of a weakly continuous Hilbert bundle Ω_{wk} relative to a continuous Hilbert bundle Ω over a locally compact Hausdorff space T. Particular cases of this notion have been previously considered in [15, 23]. It is natural to consider Ω as a C*-module over the abelian C*-algebra $C_0(T)$; if, moreover, T is a Stonean space Δ, we then show Ω_{wk} carries the structure of a faithful AW*-module over $C(\Delta)$. In this latter situation, such C*-modules are called Kaplansky–Hilbert modules. We study the C*-modules Ω and Ω_{wk}, as well as certain C*-algebras of endomorphisms of these modules, using the beautiful machinery Kaplansky developed in his seminal work from the early 1950s [16]. In particular, we prove that the C*-algebra $B(\Omega_{\mathrm{wk}})$ of bounded adjointable endomorphisms of Ω_{wk} is the injective envelope and second-order local multiplier algebra of the C*-algebra $K(\Omega)$ of "compact" endomorphisms of Ω.

Assuming that $T = \Delta$, a Stonean space, and in postponing the precise definitions until the following section, we summarise in this paragraph the main results of the paper. In Section 2, we show that Ω_{wk} is a Kaplansky–Hilbert module that contains Ω as a C*-submodule such that $\Omega^{\perp} = \{0\}$. In

Section 3, we prove that $B(\Omega_{\mathrm{wk}})$ is the injective envelope of both $K(\Omega)$ and the Fell continuous trace C*-algebra A induced by the bundle Ω. Section 4 deals with local multipliers, and we show that $B(\Omega_{\mathrm{wk}})$ is the second-order local multiplier algebra of both $K(\Omega)$ and Fell algebra A. We also prove that the equality $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = I(A)$ holds for certain type I non-separable C*-algebras, generalising a result of Somerset [21]. Finally, in Section 5 we find that a direct-sum decomposition of Ω_{wk} leads to a corresponding decomposition of (the generally non-AW*) algebra $M_{\mathrm{loc}}(A)$ but not to a decomposition of A.

1 Preliminaries

If T is a locally compact Hausdorff space and $\{H_t\}_{t\in T}$ is family of Hilbert spaces, a vector field on T with fibres H_t is a function $\nu : T \to \bigsqcup_t H_t$ in which $\nu(t) \in H_t$, for every $t \in T$. Such a vector field ν is said to be bounded if the function $t \mapsto \|\nu(t)\|$ is bounded. From this point on, the notation $T \to \bigsqcup_t H_t$ will be taken to also imply that, for all t, the point t is mapped into the corresponding fibre H_t.

Definition 1.1. *A* continuous Hilbert bundle *[8] is a triple* $(T, \{H_t\}_{t\in T}, \Omega)$, *where* Ω *is a set of vector fields on* T *with fibres* H_t *such that:*

(I) Ω *is a* $C(T)$*-module with the action* $(f \cdot \omega)(t) = f(t)\omega(t)$*;*

(II) *for each* $t_0 \in T$, $\{\omega(t_0) : \omega \in \Omega\} = H_{t_0}$*;*

(III) *the map* $t \mapsto \|\omega(t)\|$ *is continuous, for all* $\omega \in \Omega$*;*

(IV) Ω *is closed under local uniform approximation—that is, if* $\xi : T \to \bigsqcup_t H_t$ *is any vector field such that for every* $t_0 \in T$ *and* $\varepsilon > 0$ *there is an open set* $U \subset T$ *containing* t_0 *and a* $\omega \in \Omega$ *with* $\|\omega(t) - \xi(t)\| < \varepsilon$ *for all* $t \in U$*, then necessarily* $\xi \in \Omega$*.*

Dixmier and Douady [8] show that (I), (II), and (IV) can be replaced by other axioms, such as those given by Fell [10], without altering the structure that arises. For example, in the presence of the other axioms, (II) is equivalent to "$\{\omega(t_0) : \omega \in \Omega\}$ is dense in H_{t_0}, for each $t_0 \in T$"; in the presence of (IV), axiom (I) can be replaced by "Ω is a complex vector space".

We turn next to the notion of a weakly continuous Hilbert bundle. If $(T, \{H_t\}_{t\in T}, \Omega)$ is a continuous Hilbert bundle then, by the polarisation identity, the function $t \mapsto \langle\omega_1(t), \omega_2(t)\rangle$ is continuous for all $\omega_1, \omega_2 \in \Omega$. In defining $\langle\omega_1, \omega_2\rangle$ to be the map $T \to \mathbb{C}$ given by $t \mapsto \langle\omega_1(t), \omega_2(t)\rangle$, one

obtains a $C(T)$-valued inner product on Ω which gives Ω the structure of an inner product module over $C(T)$.

Definition 1.2. *A vector field $\nu : T \to \bigsqcup_t H_t$ is said to be* weakly continuous *with respect to the continuous Hilbert bundle $(T, \{H_t\}_{t\in T}, \Omega)$ if the function*

$$t \longmapsto \langle \nu(t), \omega(t)\rangle$$

is continuous for all $\omega \in \Omega$. The set of all bounded weakly continuous vector fields with respect to a given Ω will be denoted by Ω_{wk}, that is

$$\Omega_{\mathrm{wk}} = \{\nu : T \to \bigsqcup_t H_t : \ \sup_t \|\nu(t)\| < \infty \ \ \textit{and } \nu \textit{ is weakly continuous}\}.$$

We will call the quadruple $(T, \{H_t\}_{t\in T}, \Omega, \Omega_{\mathrm{wk}})$ a weakly continuous Hilbert bundle *over T.*

We remark that when T is compact, Ω_{wk} is a $C(T)$-module under the pointwise module action, and also $\Omega \subset \Omega_{\mathrm{wk}}$ (because then every continuous field on T is bounded). However, the function $t \mapsto \langle \nu_1(t), \nu_2(t)\rangle$ is generally not continuous for arbitrary $\nu_1, \nu_2 \in \Omega_{\mathrm{wk}}$. Thus, although Ω_{wk} is, algebraically, a module over $C_b(T)$, it is not in general an inner product module over $C_b(T)$. Nevertheless, if T has the right topology—namely that of a Stonean space—then we show (Theorem 2.6) that it is possible to endow a weakly continuous Hilbert bundle with the structure of a C*-module over the C*-algebra of continuous complex-valued functions on T.

The continuous trace C*-algebras we consider herein were first studied by Fell [10]. We now recall their definition.

Assume that $\{A_t\}_{t\in T}$ is a family of C*-algebras indexed by the locally compact Hausdorff topological space T. An operator field is a map $a : T \to \bigsqcup_t A_t$ such that $a(t) \in A_t$, for each $t \in T$.

Definition 1.3. *Let $(T, \{H_t\}_{t\in T}, \Omega)$ be a continuous Hilbert bundle. An operator field $a : T \to \bigsqcup_{t\in T} K(H_t)$ is:*

i. almost finite-dimensional *(with respect to Ω) if for each $t_0 \in T$ and $\varepsilon > 0$ there exist an open set $U \subset T$ containing t_0 and $\omega_1, \dots, \omega_n \in \Omega$ such that*

  (a) $\omega_1(t), \dots, \omega_n(t)$ are linearly independent for every $t \in U$, and

  (b) $\|p_t a(t) p_t - a(t)\| < \varepsilon$ for all $t \in U$, where $p_t \in B(H_t)$ is the projection with range $\mathrm{Span}\,\{\omega_j(t) : 1 \le j \le n\}$*;*

ii. weakly continuous *(with respect to* Ω*) if the complex-valued function*

$$t \longmapsto \langle a(t)\omega_1(t), \omega_2(t)\rangle$$

is continuous for every $\omega_1, \omega_2 \in \Omega$*.*

Definition 1.4. ([10]) *Let* $(T, \{H_t\}_{t\in T}, \Omega)$ *be a continuous Hilbert bundle. The* Fell algebra of the Hilbert bundle $(T, \{H_t\}_{t\in T}, \Omega)$*, denoted by* $A = A(T, \{H_t\}_{t\in T}, \Omega)$*, is the set of all weakly continuous, almost finite-dimensional operator fields* $a : T \to \bigsqcup_{t\in T} K(H_t)$ *for which* $t \mapsto \|a(t)\|$ *is continuous and vanishes at infinity, endowed with pointwise operations and norm*

$$\|a\| = \max_{t\in T} \|a(t)\|, \qquad a \in A.$$

We shall make repeated use of the following fact about the Fell algebras of Hilbert bundles: if $A = A(T, \{H_t\}_{t\in T}, \Omega)$, for some continuous Hilbert bundle $(T, \{H_t\}_{t\in T}, \Omega)$, then A is a continuous trace C*-algebra with spectrum $\hat{A} \simeq T$ [10, Theorems 4.4, 4.5].

2 An AW*-module Structure for Ω_{wk}

Assume henceforth that $T = \Delta$ is a Stonean space; that is, Δ is Hausdorff, compact, and extremely disconnected. The abelian C*-algebra $C(\Delta)$ is an AW*-algebra and so one may ask whether the C*-modules Ω and Ω_{wk} are AW*-modules in the sense of Kaplansky [16]. We shall show that this is indeed true for the module Ω_{wk}. As a consequence of this last fact we shall get that the C*-algebra $B(\Omega_{\mathrm{wk}})$ of bounded adjointable endomorphisms of Ω_{wk} is an AW*-algebra of type I.

The following lemmas are needed to describe the $C(\Delta)$-Hilbert module structure of Ω_{wk}.

Lemma 2.1. *Let* $f : \Delta \to \mathbb{R}$ *be a lower semicontinuous function such that there exist* $g \in C(\Delta)$ *and a meagre set* $M \subset \Delta$ *with* $f(s) = g(s)$ *for all* $s \in \Delta \setminus M$*. Then*

$$\sup_{s\in\Delta} g(s) = \sup_{s\in\Delta\setminus M} f(s) = \sup_{s\in\Delta} f(s).$$

Proof. Let $\rho = \sup_{s\in\Delta\setminus M} f(s) = \sup_{s\in\Delta\setminus M} g(s) \le \sup_{s\in\Delta} g(s)$; then $f(s) \le \rho$ for all $s \in \Delta \setminus M$. Because Δ is a Baire space, $\overline{\Delta \setminus M} = \Delta$; thus, by the lower semi-continuity, $f(s) \le \rho$ for every $s \in \Delta$. The same argument yields that $g(s) \le \rho$ for all $s \in \Delta$. $\square$

Lemma 2.2. *Assume that $(\Delta, \{H_s\}_{s\in\Delta}, \Omega)$ is a continuous Hilbert bundle and $\nu \in \Omega_{\mathrm{wk}}$. Then*

i. *the function $s \mapsto \|\nu(s)\|^2$ is lower semicontinuous;*

ii. *there is a meagre subset $M \subset \Delta$ and a continuous function $h : \Delta \to \mathbb{R}_+$ such that*

(a) $h(s) = \|\nu(s)\|^2$ for all $s \in \Delta \setminus M$, and

(b) $\|h\| = \sup_{s\in\Delta\setminus M} \|\nu(s)\|^2 = \sup_{s\in\Delta} \|\nu(s)\|^2$.

Proof. Let $r \in \mathbb{R}$ be fixed and consider $U_r = \{s \in \Delta : r < \|\nu(s)\|^2\}$. We aim to show that U_r is open. Choose $s_0 \in U_r$. Thus, $r < \|\nu(s_0)\|^2$. By Parseval's formula, there are orthonormal vectors $\xi_1, \dots, \xi_n \in H_{s_0}$ such that $r < \sum_{j=1}^{n} |\langle \nu(s_0), \xi_j\rangle|^2 \leq \|\nu(s_0)\|^2$. Choose any $\mu_1, \dots, \mu_n \in \Omega$ such that $\mu_j(s_0) = \xi_j$, for each j. Because $\xi_1, \dots, \xi_n$ are orthogonal, $\mu_1(s), \dots, \mu_n(s)$ are linearly independent in an open neighbourhood of s_0. Hence, by [10, Lemma 4.2], there is an open set V containing s_0 and vector fields $\omega_1, \dots, \omega_n \in \Omega$ such that $\omega_1(s), \dots, \omega_n(s)$ are orthonormal for all $s \in V$, and $\omega_j(s_0) = \xi_j$ for each j. The function

$$g(s) \;=\; \sum_{j=1}^{n} |\langle \nu(s), \omega_j(s)\rangle|^2$$

on Δ is continuous and satisfies $g(s) \leq \|\nu(s)\|^2$, for every $s \in V$, and $r < g(s_0)$. Therefore, by the continuity of g, there is an open set $W \subset V$ containing s_0 such that $r < g(s) \leq \|\nu(s)\|^2$ for all $s \in W$. This proves that U_r contains an open set around each of its points. That is, U_r is open.

Because every bounded nonnegative lower semicontinuous function on a Stonean space Δ agrees with a nonnegative continuous function off a meagre set M [24, Proposition III.1.7], the function $h \in C(\Delta)$ as in (**ii**) exists and satisfies $h(s) = \|\nu(s)\|^2$ for $s \in \Delta \setminus M$.

The last statement follows from Lemma 2.1. □

Let $(\Delta, \{H_t\}_{t\in\Delta}, \Omega, \Omega_{\mathrm{wk}})$ be a weakly continuous Hilbert bundle over Δ. Given $\nu \in \Omega_{\mathrm{wk}}$, the function h that arises in Lemma 2.2 will be denoted by $\langle \nu, \nu\rangle$. There is no ambiguity in so doing because if $h_1, h_2 \in C(\Delta)$ and if $h_1(s) = h_2(s)$ for all $s \notin (M_1 \cup M_2)$ for some meagre subsets M_1 and M_2, then h_1 and h_2 agree on Δ. (If not, then by continuity, h_1 and h_2 would

differ on an open set U; but $\emptyset \neq U \subset M_1 \cup M_2$ is in contradiction to the fact that no meagre set in a Baire space can contain a nonempty open set.)

Now use the polarisation identity to define $\langle \nu_1, \nu_2 \rangle \in C(\Delta)$ for any pair $\nu_1, \nu_2 \in \Omega_{\rm wk}$. This gives $\Omega_{\rm wk}$ the structure of pre-inner product module over $C(\Delta)$ whereby for each $\nu_1, \nu_2 \in \Omega_{\rm wk}$ there is a meagre subset $M_{\nu_1,\nu_2} \subset \Delta$ such that the continuous function $\langle \nu_1, \nu_2 \rangle$ satisfies

$$\langle \nu_1, \nu_2 \rangle (s) \;=\; \langle \nu_1(s), \nu_2(s) \rangle \,, \qquad \forall\, s \in \Delta \setminus M_{\nu_1,\nu_2} \,.$$

In particular, if $\nu \in \Omega_{\rm wk}$ and $\omega \in \Omega$, then

$$\langle \nu, \omega \rangle (s) \;=\; \langle \nu(s), \omega(s) \rangle \,, \qquad \forall\, s \in \Delta \,.$$

In fact, $\Omega_{\rm wk}$ is an inner product module over $C(\Delta)$, for if $\nu \in \Omega_{\rm wk}$ satisfies $\langle \nu, \nu \rangle = 0$, then Lemma 2.2 yields $\|\nu(s)\|^2 = 0$ for all $s \in \Delta$. Therefore,

$$\|\nu\| \;=\; \|\langle \nu, \nu \rangle\|^{1/2} \,, \quad \nu \in \Omega_{\rm wk} \,,$$

defines a norm on $\Omega_{\rm wk}$, where

$$\|\nu\|^2 \;=\; \sup_{s \in \Delta} \langle \nu(s), \nu(s) \rangle \;=\; \|\langle \nu, \nu \rangle\| \,. \tag{1}$$

Recall that given a C*-algebra B, a *Hilbert C*-module over* B is a left B-module E together with a B-valued definite sequilinear map $\langle \, , \, \rangle$ such that E is complete with the norm $\|\nu\| = \|\langle \nu, \nu \rangle\|^{1/2}$ (we refer to [17] for a detailed account on Hilbert modules).

Note that if $\nu \in \Omega_{\rm wk}$, then $|\nu|(s) := \langle \nu, \nu \rangle^{1/2}(s) \geq \|\nu(s)\|$ for $s \in \Delta$ and there exists a meagre set $M \subset \Delta$ with $|\nu|(s) = \|\nu(s)\|$ if $s \in (\Delta \setminus M)$ (Lemma 2.2). These facts will be used repeatedly from now on.

Proposition 2.3. *$\Omega_{\rm wk}$ is a C*-module over $C(\Delta)$ and Ω is a C*-submodule of $\Omega_{\rm wk}$.*

Proof. The only Hilbert C*-module axiom that is not obviously satisfied by $\Omega_{\rm wk}$ is the axiom of completeness. Let $\{\nu_i\}_{i \in \mathbb{N}}$ be a Cauchy sequence in $\Omega_{\rm wk}$. By the equality (1), $\{\nu_i(s)\}_{i \in \mathbb{N}}$ is a Cauchy sequence in H_s for every $s \in \Delta$. Let $\nu(s) \in H_s$ denote the limit of this sequence so that $\nu : \Delta \to \bigsqcup_{s \in \Delta} H_s$ is a vector field.

Choose $\omega \in \Omega$ and consider the function $g_{i,\omega} \in C(\Delta)$ given by $g_{i,\omega}(s) = \langle \omega(s), \nu_i(s) \rangle$. Let $\varepsilon > 0$. Then there is $N_\varepsilon \in \mathbb{N}$ such that $\|\nu_i - \nu_j\| < \varepsilon$, for all $i, j \geq N_\varepsilon$. Therefore, the Cauchy-Schwarz inequality yields

$$\sup_{s \in \Delta} |g_{i,\omega}(s) - g_{j,\omega}(s)| \;<\; \varepsilon\, \|\omega\| \,, \quad \forall\, i, j \geq N_\varepsilon \,.$$

Thus, the sequence $\{g_{i,\omega}\}_i$ is Cauchy in $C(\Delta)$; let $g_\omega \in C(\Delta)$ denote its limit. Observe that $g_\omega(s) = \lim_i \langle \nu_i(s), \omega(s)\rangle = \langle \nu(s), \omega(s)\rangle$, for all $s \in \Delta$. As the choice of $\omega \in \Omega$ is arbitrary, this shows that ν is weakly continuous. The Cauchy sequence $\{\nu_i\}_{i\in\mathbb{N}}$ is necessarily uniformly bounded by, say, $\rho > 0$, and then $\|\nu(s)\| \le \rho$ for every $s \in \Delta$. That is, the function $s \to \|\nu(s)\|$ is bounded and so $\nu \in \Omega_{\rm wk}$. Finally, if $i, j \ge N_\varepsilon$, then for any $s \in \Delta$ we have $\|\nu(s) - \nu_i(s)\| \le \|\nu(s) - \nu_j(s)\| + \|\nu_j(s) - \nu_i(s)\| \le \|\nu(s) - \nu_j(s)\| + \varepsilon$, and so letting $j \to \infty$ yields $\|\nu(s) - \nu_i(s)\| \le \varepsilon$ for every $s \in \Delta$. That is, $\|\nu - \nu_i\| \to 0$, which proves that $\Omega_{\rm wk}$ is complete.

For the case of Ω, let $\{\omega_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in Ω. For each $s \in \Delta$, $\{\omega_n(s)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in H_s; let $\omega(s)$ denote the limit. Since the limit is uniform, it is in particular locally uniform, and so $\omega \in \Omega$. Hence, Ω is complete. □

Definition 2.4. *A Hilbert C^*-module E over a C^*-algebra B is called a* Kaplansky–Hilbert module *if in addition B is an abelian AW^*-algebra and the following three properties hold [16, p. 842] (Kaplansky's original term for such a module was "faithful AW^*-module"):*

i. *if $e_i \cdot \nu = 0$ for some family $\{e_i\}_i \subset B$ of pairwise-orthogonal projections and $\nu \in E$, then also $e \cdot \nu = 0$, where $e = \sup_i e_i$;*

ii. *if $\{e_i\}_i \subset B$ is a family of pairwise-orthogonal projections such that $1 = \sup_i e_i$, and if $\{\nu_i\}_i \subset E$ is a bounded family, then there is a $\nu \in E$ such that $e_i \cdot \nu = e_i \cdot \nu_i$ for all i;*

iii. *if $\nu \in E$, then $g \cdot \nu = 0$ for all $g \in B$ only if $\nu = 0$.*

Remark 2.5. *The element $\nu \in E$ obtained in the situation described in (ii) will sometimes be denoted as $\sum_i e_i \nu_i$. It should be emphasized that this is not a pointwise sum.*

Theorem 2.6. *$\Omega_{\rm wk}$ is a Kaplansky–Hilbert module over $C(\Delta)$.*

Proof. For property (i), assume that $\nu \in \Omega_{\rm wk}$ and $\{e_i\}_i \subset C(\Delta)$ is a family of pairwise-orthogonal projections with supremum $e \in C(\Delta)$ for which $e_i \cdot \nu = 0$ for all i. Because projections in $C(\Delta)$ are the characteristic functions of clopen sets, there are pairwise-disjoint clopen sets $U_i \subset \Delta$ such that $e_i = \chi_{U_i}$. Thus, for each i, using Lemma 2.2,

$$\begin{aligned} 0 = \|e_i \cdot \nu\|^2 &= \max_{s\in\Delta} \langle e_i \cdot \nu, e_i \cdot \nu\rangle(s) = \sup_{s\in\Delta} \langle e_i(s)\nu(s), e_i(s)\nu(s)\rangle \\ &= \max_{s\in\Delta} e_i(s)\ [\langle \nu, \nu\rangle(s)] = \max_{s\in U_i} \langle \nu, \nu\rangle(s)\,, \end{aligned}$$

and so $\langle\nu,\nu\rangle(s)=0$ for every $s\in U_i$. Let $U=\bigcup_i U_i$. The set $\overline{U}$ is clopen and $\chi_{\overline{U}}=\sup_i\, e_i=e$ [5, §8]. As $\langle\nu,\nu\rangle$ is a continuous function that vanishes on U, it also vanishes on $\overline{U}$. Hence,

$$\|e\cdot\nu\|^2\;=\;\max_{s\in\Delta}\,e(s)\,[\langle\nu,\nu\rangle(s)]\;=\;\max_{s\in\overline{U}}\,\langle\nu,\nu\rangle(s)\;=\;0\,,$$

which yields property (i).

For the proof of property (ii), assume that $\{e_i\}_i\subset C(\Delta)$ is a family of pairwise-orthogonal projections such that $1=\sup_i\, e_i$ and that $\{\nu_i\}_i\subset\Omega_{\rm wk}$ is a family such that $K=\sup\|\nu_i\|<\infty$; we aim to prove that there is a $\nu\in\Omega_{\rm wk}$ such that $e_i\cdot\nu=e_i\cdot\nu_i$ for all i. As before, assume that $e_i=\chi_{U_i}$ and $U=\bigcup_i U_i$. Then $1=\sup_i\, e_i$ implies that $\overline{U}=\Delta$.

For each $\omega\in\Omega$, consider the unique function $f_\omega\in C(\Delta)$ such that $e_i\,f_\omega=e_i\,\langle\omega,\nu_i\rangle$ for all i (its existence guaranteed by the fact that Δ is the Stone–Čech compactification of U). Note that for $s\in U_i$ we have that $f_\omega(s)=\langle\omega(s),\nu_i(s)\rangle$. Hence, $|f_\omega(s)|\le K\,\|\omega(s)\|$ for $s\in U$; the same inequality holds for all $s\in\Delta$ because $\overline{U}=\Delta$ and both sides of the inequality are continuous functions of s. Moreover, if $\omega_1,\,\omega_2\in\Omega$ and $\alpha\in\mathbb{C}$ then, for $s\in U$ we get that $f_{\alpha\,\omega_1+\omega_2}(s)=\alpha\,f_{\omega_1}(s)+f_{\omega_2}(s)$ and, therefore, that $f_{\alpha\,\omega_1+\omega_2}=\alpha\,f_{\omega_1}+f_{\omega_2}$. Thus, for each $s\in\Delta$ the function $\omega(s)\mapsto f_\omega(s)$ is a well-defined, bounded linear functional on H_s. Let $\nu(s)\in H_s$ be the representing vector for this functional, yielding a vector field $\nu:\Delta\to\bigsqcup_{s\in\Delta}H_s$. Since $\langle\nu(s),\omega(s)\rangle=\overline{f_\omega(s)}$, for every $\omega\in\Omega$, ν is weakly continuous. It remains to show that ν is a bounded vector field. If $s\in U$,

$$\|\nu(s)\|=\sup_{\omega\in\Omega,\|\omega(s)\|=1}|\langle\omega(s),\nu(s)\rangle|=\sup_{\omega\in\Omega,\|\omega(s)\|=1}|f_\omega(s)|\le\sup_i\|\nu_i\|=K\,,$$

which shows that $\|\nu(s)\|$ is uniformly bounded on U. Thus, since U is dense, the lower semicontinuous function $s\mapsto\|\nu(s)\|^2$ is bounded on Δ. Therefore, $\nu\in\Omega_{\rm wk}$.

Now we show that $e_i\cdot\nu=e_i\cdot\nu_i$, for all i. Fix i and $s\in U_i$ and consider $\omega\in\Omega$. Then,

$$\begin{aligned}\langle\omega(s),\,e_i(s)\,\nu(s)\rangle&=\langle\omega(s),\,\nu(s)\rangle=f_\omega(s)\\&=e_i(s)\,f_\omega(s)=e_i(s)\langle\omega(s),\nu_i(s)\rangle\\&=\langle\omega(s),e_i(s)\,\nu_i(s)\rangle\,.\end{aligned}$$

Since $(e_i\cdot\nu)(s)=0=(e_i\cdot\nu_i)(\mathrm{s})$ for $s\in\Delta\setminus U_i$ we conclude that $e_i\cdot\nu=e_i\cdot\nu_i$.

For the proof of property (iii), assume that $\nu\in\Omega_{\rm wk}$ satisfies $g\cdot\nu=0$ for all $g\in C(\Delta)$. Then, in particular, $\langle\nu,\nu\rangle\cdot\nu=0$, so $\langle\nu,\nu\rangle=0$. Hence, from $\|\nu\|=\|\langle\nu,\nu\rangle\|^{1/2}=0$ we conclude that $\nu=0$. $\square$

3 Endomorphisms of Ω and $\Omega_{\rm wk}$

Throughout this section A will denote the Fell C*-algebra of the continuous Hilbert bundle $(\Delta, \{H_s\}_{s\in\Delta}, \Omega)$, as described in Definition 1.4, with Δ Stonean. Let $B(\Omega)$ and $B(\Omega_{\rm wk})$ denote, respectively, the C*-algebras of adjointable $C(\Delta)$-endomorphisms of Ω and $\Omega_{\rm wk}$. Since, by Theorem 2.6, $\Omega_{\rm wk}$ is a Kaplansky–Hilbert AW*-module over $C(\Delta)$, $B(\Omega_{\rm wk})$ coincides with the set of all $C(\Delta)$-endomorphisms of $\Omega_{\rm wk}$ [16, Theorem 6] and is a type I AW*-algebra with centre $C(\Delta)$ [16, Theorem 7].

In the particular case where Ω is given by the trivial Hilbert bundle $(\Delta, \{H\}_{s\in\Delta}, C(\Delta, H))$ with H is a fixed Hilbert space, Hamana [15] proved that $B(\Omega_{\rm wk}) \cong C(\Delta)\overline{\otimes}B(H)$, the monotone complete tensor product of $C(\Delta)$ and $B(H)$.

For each $\nu_1, \nu_2 \in \Omega_{\rm wk}$, consider the endomorphism Θ_{ν_1,ν_2} on $\Omega_{\rm wk}$ defined by

$$\Theta_{\nu_1,\nu_2}(\nu) = \langle \nu, \nu_2\rangle \cdot \nu_1 , \quad \nu \in \Omega_{\rm wk} .$$

For a Hilbert bundle Ω_0, let

$$F(\Omega_0) = \left\{ \sum_{j=1}^{n} \Theta_{\omega_j,\omega_j'} \, : \, n \in \mathbb{N},\, \omega_j, \omega_j' \in \Omega \right\}.$$

We will consider both $F(\Omega)$ and $F(\Omega_{\rm wk})$.

If $\omega_1, \omega_2 \in \Omega$, then $\Theta_{\omega_1,\omega_2}(\omega) \in \Omega$ for all $\omega \in \Omega$, and so $F(\Omega) \subset B(\Omega)$. In fact, $F(\Omega)$ and $F(\Omega_{\rm wk})$ are algebraic ideals in $B(\Omega)$ and $B(\Omega_{\rm wk})$ respectively. The norm-closures of these algebraic ideals, namely $K(\Omega)$ and $K(\Omega_{\rm wk})$, are essential ideals in each of $B(\Omega)$ and $B(\Omega_{\rm wk})$—called the ideals of compact endomorphisms—and the multiplier algebras of $K(\Omega)$ and $K(\Omega_{\rm wk})$ are, respectively, $B(\Omega)$ and $B(\Omega_{\rm wk})$ [17].

When referring to rank-1 operators x acting on a Hilbert space H, we will use the notation $x = \xi \otimes \eta$ for such an operator—the action on $\gamma \in H$ given by $\gamma \mapsto \langle \gamma, \eta\rangle\, \xi$—and we reserve the notation $\Theta_{\xi,\eta}$ for "rank-1" operators acting on a Hilbert module.

The term "homomorphism" will be used to mean a $*$-homomorphism between C*-algebras.

For any C*-algebra B, we denote the injective envelope [13], [18, Chapter 15] of B by $I(B)$ (and we consider $I(B)$ as a C*-algebra rather than as an operator system).

The main result of the present section is the following.

Theorem 3.1. *There exist C^*-algebra embeddings such that*

$$K(\Omega) \subset A \subset B(\Omega) \subset B(\Omega_{\rm wk}) = I(K(\Omega)). \qquad (2)$$

In particular, $I(K(\Omega)) = I(A) = I(B(\Omega)) = B(\Omega_{\rm wk})$.

The proof of Theorem 3.1 and a description of the inclusions in (2) begin with the following set of results.

Lemma 3.2. *For every $a \in A$ and $\omega \in \Omega$, the vector field $a \cdot \omega$ defined by $a \cdot \omega(s) = a(s)\omega(s)$ is an element of Ω.*

Proof. Let $a \in A$. Then $a^*a \in A_+$ and since all fields in A are weakly continuous, for every $\omega \in \Omega$ the map $s \mapsto \|a(s)\omega(s)\| = \langle a^*a \cdot \omega(s), \omega(s)\rangle^{1/2}$ is continuous.

Suppose $s_0 \in \Delta$ and $\varepsilon > 0$. Because $H_{s_0} = \{\mu(s_0) : \mu \in \Omega\}$, there is a $\mu \in \Omega$ such that $a(s_0)\omega(s_0) = \mu(s_0)$. Since

$$\|a \cdot \omega(s) - \mu(s)\|^2 = \|a(s)\omega(s)\|^2 + \|\mu(s)\|^2 - 2\mathrm{Re}\,\langle a(s)\omega(s), \mu(s)\rangle$$

is continuous on Δ and vanishes at s_0, there is an open set $U \subset \Delta$ containing s_0 such that $\|a \cdot \omega(s) - \mu(s)\| < \varepsilon$ for all $s \in U$. As Ω is closed under local uniform approximation, this proves that $a \cdot \omega \in \Omega$. □

Proposition 3.3. *The map $\varrho : A \to B(\Omega)$ given by $\varrho(a)\omega = a \cdot \omega$, for $a \in A$ and $\omega \in \Omega$ is an isometric homomorphism. Furthermore, $K(\Omega) \subset \varrho(A) \subset B(\Omega)$ as C^*-algebras.*

Proof. It is clear that ϱ is a homomorphism, and so we only need to verify that it is one-to-one. To this end, assume that $\varrho(a) = 0$. Thus, $a(s)\omega(s) = 0$ for every $\omega \in \Omega$ and every $s \in \Delta$. Because $H_s = \{\omega(s) : \omega \in \Omega\}$, this implies that $a(s) = 0$ for all $s \in \Delta$, and so $a = 0$.

To show $K(\Omega) \subset \varrho(A) \subset B(\Omega)$ as C*-algebras, consider $\Theta_{\omega_1,\omega_2}$ with $\omega_1, \omega_2 \in \Omega$. The map $s \mapsto \|\Theta_{\omega_1(s),\omega_2(s)}\|$ is continuous because $\|\Theta_{\omega_1(s),\omega_2(s)}\| = \|\omega_1(s)\|\,\|\omega_2(s)\|$. For any $\eta_1, \eta_2 \in \Omega$, the map

$$\langle \Theta_{\omega_1,\omega_2} \cdot \eta_1, \eta_2\rangle(s) = \langle \eta_1, \omega_2\rangle(s)\,\langle \omega_1, \eta_2\rangle(s) = \langle \eta_1(s), \omega_2(s)\rangle\,\langle \omega_1(s), \eta_2(s)\rangle$$

is continuous. So $\Theta_{\omega_1,\omega_2}$ is also finite dimensional and weakly continuous, which shows that $\Theta_{\omega_1,\omega_2} \in A$ and $K(\Omega) \subset \varrho(A)$. □

Lemma 3.4. *With respect to the inclusion $\Omega \subset \Omega_{\rm wk}$, we have $\Omega^\perp = \{0\}$.*

Proof. Let $\nu \in \Omega_{\mathrm{wk}}$ be such that $\langle \nu, \omega \rangle = 0$, for every $\omega \in \Omega$. That is, for every $\omega \in \Omega$ and for every $s \in \Delta$, $\langle \nu(s), \omega(s) \rangle = 0$. If $\nu \neq 0$, there exists $s_0 \in \Delta$ such that $\nu(s_0) \neq 0$. By axiom (II) in Definition 1.1, there exists $\omega \in \Omega$ such that $\omega(s_0) = \nu(s_0)$, in contradiction to $\langle \nu(s_0), \omega(s_0) \rangle = 0$. $\square$

Lemma 3.5. *If $t_0 \in \Delta$ and $\xi \in H_{t_0}$, then there exists $\omega \in \Omega$ such that $\omega(t_0) = \xi$ and $\|\omega\| = \|\xi\|$.*

Proof. The case $\xi = 0$ is trivial. So assume that $\|\xi\| > 0$. Let $\omega' \in \Omega$ with $\omega'(t_0) = \xi$. Fix a clopen neighbourhood V of t_0 such that $V \subset \{t \in T : \|\omega'(t)\| \geq \|\omega'(t_0)\|/2\}$. Let $h'(\cdot) = \|\xi\| \cdot \|\omega'(\cdot)\|^{-1} \in C(V)$; then h' extends to a continuous function $h \in C(\Delta)$ with $h|_{\Delta \setminus V} = 0$. It is now straightforward to show that $\omega = h \cdot \omega' \in \Omega$ has the desired properties. $\square$

Proposition 3.6. *There exists an isometric homomorphism $\vartheta : B(\Omega) \to B(\Omega_{\mathrm{wk}})$ such that for $a \in A,\ \nu \in \Omega_{\mathrm{wk}}$,*

$$(\vartheta(\varrho(a))\nu)(s) = a(s)\nu(s), \quad s \in \Delta\,. \tag{3}$$

Proof. Assume that $b \in B(\Omega)$ and $\omega \in \Omega$, $s \in \Delta$. By Lemma 3.5,

$$\begin{aligned}
\|(b\,\omega)(s)\| &= \sup_{\xi \in H_s,\, \|\xi\|=1} |\langle (b\,\omega)(s), \xi \rangle| = \sup_{\eta \in \Omega,\, \|\eta\|=1} |\langle (b\,\omega)(s), \eta(s) \rangle| \\
&= \sup_{\eta \in \Omega,\, \|\eta\|=1} |\langle b\,\omega, \eta \rangle(s)| = \sup_{\eta \in \Omega,\, \|\eta\|=1} |\langle \omega(s), (b^*\eta)(s) \rangle| \\
&\leq \|\omega(s)\| \sup_{\eta \in \Omega,\, \|\eta\|=1} \|b^*\eta\| \leq \|\omega(s)\|\, \|b^*\| = \|\omega(s)\|\, \|b\|\,.
\end{aligned}$$

Therefore the function $\omega(s) \mapsto (b\,\omega)(s)$ is well defined and induces a bounded linear operator $b(s) \in B(H_s)$ such that $(b\,\omega)(s) = b(s)\,\omega(s)$, for $s \in \Delta$ and $\omega \in \Omega$, with $\sup_{s \in \Delta} \|b(s)\| \leq \|b\|$. Moreover,

$$\begin{aligned}
\|b\| &= \sup_{\|\omega\|=1} \|b \cdot \omega\| = \sup_{\|\omega\|=1} \sup_{s} \|b \cdot \omega(s)\| = \sup_{\|\omega\|=1} \sup_{s} \|b(s)\omega(s)\| \\
&\leq \sup_{\|\omega\|=1} \sup_{s} \|b(s)\|\, \|\omega(s)\| \leq \sup_{s} \|b(s)\| \leq \|b\|\,,
\end{aligned}$$

and so $\sup_{s \in \Delta} \|b(s)\| = \|b\|$. Suppose now that $\nu \in \Omega_{\mathrm{wk}}$ and $s \in \Delta$, and define a vector field $\vartheta b \nu$ by $(\vartheta b\,\nu)(s) = b(s)\,\nu(s)$. If $\eta \in \Omega$, then

$$\langle (\vartheta b\,\nu)(s), \eta(s) \rangle \;=\; \langle \nu(s), b(s)^*\eta(s) \rangle \;=\; \langle \nu(s), (b^*\eta)(s) \rangle$$

is continuous, which shows that $\vartheta b\,\nu$ is weakly continuous with respect to Ω. Since $\vartheta b\,\nu$ is also uniformly bounded, we conclude that $\vartheta b\,\nu \in \Omega_{\mathrm{wk}}$.

It is straightforward to show that the map $\nu \mapsto \vartheta b\,\nu$ is a bounded $C(\Delta)$-endomorphism of $\Omega_{\rm wk}$ and hence it gives rise to an element $\vartheta b \in B(\Omega_{\rm wk})$. It is clear that ϑ is a homomorphism. If $\vartheta b = 0$, then $b(s)\omega(s) = 0$ for all $\omega \in \Omega$, $s \in \Delta$ and so $b(s) = 0$ for all s; then $\|b\| = \sup_s \|b(s)\| = 0$, and $b = 0$. So ϑ is one-to-one, and thus isometric. Finally, it is clear that (3) holds by construction. □

One consequence of the proof of Proposition 3.6 is that for every $b \in B(\Omega)$ there exists an operator field $\{b(s)\}_{s\in\Delta}$ acting on the Hilbert bundle $\{H_s\}_{s\in\Delta}$ such that $(b\,\omega)(s) = b(s)\,\omega(s)$, for every $s \in \Delta$. This property, however, is not shared by all elements of $B(\Omega_{\rm wk})$.

Lemma 3.7. *If $z \in B(\Omega_{\rm wk})$ and $\Theta_{\omega,\omega} z \Theta_{\mu,\mu} = 0$ for all $\omega, \mu \in \Omega$, then $z = 0$.*

Proof. For any ξ, ω, $\mu \in \Omega$ we have that

$$0 = \Theta_{\omega,\omega}\, z\, \Theta_{\mu,\mu}\, \xi = \langle \xi, \mu\rangle\, \langle z\mu, \omega\rangle\, \omega.$$

Hence, we get that

$$0 = \langle \xi, \mu\rangle\, |\langle z\mu, \omega\rangle|^2 = \langle \xi, \mu\rangle\, |\langle \mu, z^*\omega\rangle|^2.$$

We are free to choose $\xi, \mu \in \Omega$. Fix s, and choose μ with $\mu(s) = z^*\omega(s)$; let $\xi = \mu$. Then, as $\mu \in \Omega$, we get $0 = \langle \mu, \mu\rangle(s) = \langle \mu(s), \mu(s)\rangle$, so $z^*\omega(s) = \mu(s) = 0$. As $s \in \Delta$ is arbitrary, $z^*\omega = 0$ for every $\omega \in \Omega$. For any $\nu \in \Omega_{\rm wk}$ and every $\omega \in \Omega$, $\langle z\nu, \omega\rangle = \langle \nu, z^*\omega\rangle = 0$. By Lemma 3.4 we conclude that $z\nu = 0$ for $\nu \in \Omega_{\rm wk}$ and hence $z = 0$. □

Proof of Theorem 3.1. We consider the embeddings $A \xrightarrow{\varrho} B(\Omega)$ and $B(\Omega) \xrightarrow{\vartheta} B(\Omega_{\rm wk})$ defined in Propositions 3.3 and 3.6. In this way, we get the inclusions in (2).

Because $B(\Omega_{\rm wk})$ is a type I AW*-algebra, it is injective [14, Proposition 5.2]. To show that $B(\Omega_{\rm wk})$ is the injective envelope $I(K(\Omega))$ of $K(\Omega)$, we need to show that the embedding $\vartheta \circ \varrho$ of $K(\Omega)$ into $B(\Omega_{\rm wk})$ is rigid [18, Theorem 15.8]: that is, we aim to prove that if $\phi : B(\Omega_{\rm wk}) \to B(\Omega_{\rm wk})$ is a unital completely positive linear map for which $\phi|_{K(\Omega)} = {\rm id}_{K(\Omega)}$, then $\phi = {\rm id}_{B(\Omega_{\rm wk})}$.

Let $\phi : B(\Omega_{\rm wk}) \to B(\Omega_{\rm wk})$ be such a ucp map with $\phi|_{K(\Omega)} = {\rm id}_{K(\Omega)}$. Suppose that $z \in B(\Omega_{\rm wk})$ and $\omega, \mu \in \Omega$. Then $\Theta_{\omega,\omega} z \Theta_{\mu,\mu} = \Theta_{\langle z\mu,\omega\rangle\omega,\mu} \in K(\Omega)$. Because $K(\Omega)$ is in the multiplicative domain of ϕ, we have that $\phi(axb) = a\phi(x)b$ for all $x \in B(\Omega_{\rm wk})$ and $a, b \in K(\Omega)$. This implies that

$$\Theta_{\omega,\omega}\phi(z)\Theta_{\mu,\mu} = \phi(\Theta_{\omega,\omega} z \Theta_{\mu,\mu}) = \phi(\Theta_{\langle z\mu,\omega\rangle\omega,\mu}) = \Theta_{\langle z\mu,\omega\rangle\omega,\mu} = \Theta_{\omega,\omega} z \Theta_{\mu,\mu},$$

and so $\Theta_{\omega,\omega}(z-\phi(z))\Theta_{\mu,\mu} = 0$. Since ω, μ were arbitrary, Lemma 3.7 implies that $z - \phi(z) = 0$ and so $\phi = \mathrm{id}_{B(\Omega_{\mathrm{wk}})}$.

We have shown above that the inclusion $K(\Omega) \subset B(\Omega_{\mathrm{wk}})$ is rigid. Moreover, $K(\Omega)$ is an essential ideal of $B(\Omega)$ and $K(\Omega) \subset A \subset B(\Omega)$. Hence, $I(K(\Omega)) = I(A) = I(B(\Omega)) = B(\Omega_{\mathrm{wk}})$. □

We conclude this section with a remark about the ideal $K(\Omega_{\mathrm{wk}})$ of $B(\Omega_{\mathrm{wk}})$. In type I AW*-algebras, the ideal generated by the abelian projections has a prominent role. As it happens, $K(\Omega_{\mathrm{wk}})$ is precisely this ideal.

Proposition 3.8. *The C*-algebra $K(\Omega_{\mathrm{wk}})$ coincides with the ideal $J \subset B(\Omega_{\mathrm{wk}})$ generated by the abelian projections of $B(\Omega_{\mathrm{wk}})$. So $K(\Omega_{\mathrm{wk}})$ is a liminal C*-algebra with Hausdorff spectrum.*

Proof. By [16, Lemma 13], a projection $e \in B(\Omega_{\mathrm{wk}})$ is abelian if and only if there exists $\nu \in \Omega_{\mathrm{wk}}$ such that $|\nu|$ is a projection in $C(\Delta)$ and $e = \Theta_{\nu,\nu}$. Hence, $J \subset K(\Omega_{\mathrm{wk}})$.

To show that $K(\Omega_{\mathrm{wk}}) \subset J$, assume $\nu \in \Omega_{\mathrm{wk}}$ is nonzero. Let $\varepsilon > 0$. We will show that there is an $x_\varepsilon \in J$ such that $\|\Theta_{\nu,\nu} - x_\varepsilon\| < \varepsilon$. Let $V \subset \Delta$ be the (clopen) closure of $\{s \in \Delta : \ |\nu|(s) < \varepsilon^{1/2}\}$, $U = \Delta \setminus V$ (also clopen) and let $g = (1/|\nu|)\,\chi_U \in C(\Delta)_+$. Then $g|\nu| = \chi_U$ and $\|\,\chi_{\Delta\setminus U}|\nu|\,\| < \varepsilon^{1/2}$. Let $\nu' = g \cdot \nu$ so that $|\nu'| = \chi_U$. Hence, $\Theta_{\nu',\nu'} \in J$ and $\Theta_{\nu',\nu'} = g^2 \cdot \Theta_{\nu,\nu}$. Let $x_\varepsilon = |\nu|^2 \cdot \Theta_{\nu',\nu'} \in J$. Then

$$x_\varepsilon = |\nu|^2 \cdot \Theta_{\nu',\nu'} = |\nu|^2\, g^2\, \Theta_{\nu,\nu} = \chi_U\, \Theta_{\nu,\nu},$$

and $x_\varepsilon - \Theta_{\nu,\nu} = \chi_{\Delta\setminus U} \cdot \Theta_{\nu,\nu}$. Then

$$\begin{aligned}
\|x_\varepsilon - \Theta_{\nu,\nu}\| &= \sup_{\eta\in(\Omega_{\mathrm{wk}})_1} \|\chi_{\Delta\setminus U} \cdot \Theta_{\nu,\nu}\,\eta\| = \sup_{\eta\in(\Omega_{\mathrm{wk}})_1} \|\chi_{\Delta\setminus U} \cdot \langle \eta,\nu\rangle\,\nu\| \\
&= \sup_{\eta\in(\Omega_{\mathrm{wk}})_1} \max_{s\in\Delta\setminus U} |\langle \eta,\nu\rangle(s)|\,\|\nu(s)\| \\
&\le \sup_{\eta\in(\Omega_{\mathrm{wk}})_1} \max_{s\in\Delta\setminus U} |\eta|(s)\,|\nu|(s)|\,\|\nu(s)\| \le \max_{s\in\Delta\setminus U} |\nu|(s)^2 < \varepsilon.
\end{aligned}$$

As ε was arbitrary and J is closed, we conclude that $\Theta_{\nu,\nu} \in J$. The polarisation identity then shows that $\Theta_{\nu_1,\nu_2} \in J$ for all $\nu_1, \nu_2 \in \Omega_{\mathrm{wk}}$. Hence, $F(\Omega_{\mathrm{wk}}) \subset J$, and so $K(\Omega_{\mathrm{wk}}) \subset J$.

It remains to justify the last assertion in the statement. By the main result of [12], the ideal generated by the abelian projections in a type I AW*-algebra is liminal and has Hausdorff spectrum. Hence, this is true of $K(\Omega_{\mathrm{wk}})$. □

4 Multiplier and Local Multiplier Algebras

In the previous section we established the inclusions $K(\Omega) \subset A \subset B(\Omega) \subset B(\Omega_{\mathrm{wk}})$, as C*-subalgebras, and we showed that $I(A) = B(\Omega_{\mathrm{wk}})$. The present section refines these inclusions to incorporate multiplier algebras and local multiplier algebras.

Given a C*-algebra C, we denote by $M(C)$ and $M_{\mathrm{loc}}(C)$ its multiplier and local multiplier algebra [2] respectively.

The second order local multiplier algebra of C is $M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(C)\right)$, the local multiplier algebra of $M_{\mathrm{loc}}(C)$. By [11, Corollary 4.3], the local multiplier algebras (of all orders) of C are C*-subalgebras of the injective envelope $I(C)$ of C. In particular, $C \subset M_{\mathrm{loc}}(C) \subset M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(C)\right) \subset I(C)$ as C*-subalgebras.

By a well known theorem of Kasparov [2, Theorem 1.2.33], [17, Theorem 2.4], $M(K(\Omega)) = B(\Omega)$. We remark that all the subalgebras we consider are essential in $B(\Omega_{\mathrm{wk}})$ (i.e. the annihilator is zero), and so whenever we write $M(C)$ for one of these subalgebras $C \subset B(\Omega_{\mathrm{wk}})$, we mean the concrete realization [20]

$$M(C) = \{x \in B(\Omega_{\mathrm{wk}}) :\ xC + Cx \subset C\}.$$

The following theorem is the main result of this section.

Theorem 4.1. *With the notations from the previous sections, we have the equality* $M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(K(\Omega))$ *and the following inclusions (as C*-subalgebras):*

$$\begin{aligned} M(A) &\subset M(K(\Omega)) = B(\Omega) \\ &\subset M_{\mathrm{loc}}(K(\Omega)) \subset M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(K(\Omega))\right) = B(\Omega_{\mathrm{wk}}) . \end{aligned} \tag{4}$$

In particular, $M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(A)\right) = I(A)$.

Ara and Mathieu have presented examples of Stonean spaces Δ and trivial Hilbert bundles Ω where the inclusion $M_{\mathrm{loc}}(K(\Omega)) \subset M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(K(\Omega))\right)$ in (4) is proper [3, Theorem 6.13]. As a consequence of Theorem 4.1 and the fact that $B(\Omega_{\mathrm{wk}}) = I(K(\Omega))$, we see that this gap cannot occur for higher local multiplier algebras, i.e. for all $k \geq 2$, $M_{\mathrm{loc}}^{k+1}(K(\Omega)) = M_{\mathrm{loc}}^{k}(K(\Omega))$ — where $M_{\mathrm{loc}}^{k+1}(K(\Omega)) = M_{\mathrm{loc}}(M_{\mathrm{loc}}^{k}(K(\Omega)))$ for $k \geq 1$.

The proof of Theorem 4.1 is achieved through a number of lemmas.

Lemma 4.2. *The set*

$$F_+ = \{\sum_{j=1}^{n} \Theta_{\omega_j,\omega_j} \ : \ n \in \mathbb{N},\, \omega_j \in \Omega\}$$

is dense in the positive cone of $K(\Omega)$.

Proof. Assume that $h \in K(\Omega)_+$ and let $\varepsilon > 0$ be arbitrary. For each $s_0 \in \Delta$ consider the positive compact operator $h(s_0) \in K(H_{s_0})$. Then there are vectors $\xi_1, \dots, \xi_{n_{s_0}} \in H_{s_0}$ such that

$$\|h(s_0) - \sum_{j=1}^{n_{s_0}} \xi_j \otimes \xi_j\| < \varepsilon\,.$$

Using (II) in Definition 1.1, choose $\omega_1, \dots, \omega_{n_{s_0}} \in \Omega$ such that $\omega_j(s_0) = \xi_j$, $1 \le j \le n_{s_0}$, and let $\kappa_{s_0} = \sum_{j=1}^{n_{s_0}} \Theta_{\omega_j,\omega_j}$. By continuity of the operator fields in A, there is an open set $U_{s_0} \subset \Delta$ containing s_0 such that $\|h(s) - \kappa_{s_0}(s)\| < \varepsilon$ for all $s \in U_{s_0}$.

This procedure leads to an open cover $\{U_s\}_{s\in\Delta}$ of Δ, from which (by compactness) there exists a finite subcover $\{U_1, \dots, U_m\}$ and corresponding fields $\kappa_i = \sum_{j=1}^{n_i} \Theta_{\omega_j^{[i]},\omega_j^{[i]}}$. Let $\{\psi_1, \dots, \psi_m\} \subset C(\Delta)$ be a partition of unity subordinate to $\{U_1, \dots, U_m\}$ and note that $\psi_i \cdot \Theta_{\omega_j^{[i]},\omega_j^{[i]}} = \Theta_{\psi_i^{1/2}\cdot\omega_j^{[i]},\psi_i^{1/2}\cdot\omega_j^{[i]}}$ for all j and i. Hence, the field $\kappa = \sum_{i=1}^{m} \psi_i \cdot \kappa_i$ is in F_+, and for each $s \in \Delta$,

$$\|h(s) - \kappa(s)\| = \|\sum_{i=1}^{m} \psi_i \cdot (h - \kappa_i)(s)\| \le \sum_{i=1}^{m} \psi_i(s)\|(h - \kappa_i)(s)\| < \varepsilon\,.$$

Hence, h is in the norm-closure of F_+. □

Lemma 4.3. *Let* $\{U_i\}_{i\in\Lambda}$ *be a family of pairwise disjoint clopen subsets of* Δ *whose union* U *is dense in* Δ*, and let* $c_i = \chi_{U_i} \in C(\Delta)$*, for each* $i \in \Lambda$*. Suppose that* $\{\omega_i\}_{i\in\Lambda}$ *is any bounded family in* Ω *and let* $\tilde{\omega} = \sum_{i\in\Lambda} c_i\,\omega_i \in \Omega_{\mathrm{wk}}$*, in the sense of Remark 2.5. If* $f \in C(\Delta)$ *is such that* $f(s) = 0$ *for* $s \in \Delta \setminus U$*, then* $f \cdot \tilde{\omega} \in \Omega$.

Proof. Fix $s_0 \in \Delta$ and let $\varepsilon > 0$. If $s_0 \in \Delta \setminus U$, then by the continuity of f and the fact that $f(s_0) = 0$ there exists an open subset $U_{s_0} \subset \Delta$ containing s_0 such that $|f(s)| < \varepsilon\|\tilde{\omega}\|^{-1}$ for all $s \in U_{s_0}$. Hence, the vector field $f \cdot \tilde{\omega}$ is within ε of the zero vector field $0 \in \Omega$ on the open set U_{s_0}.

On the other hand, if $s_0 \in U$, then there exists $j \in \Lambda$ such that $s_0 \in U_j$. By construction, $c_j \cdot \tilde{\omega} = c_j \cdot \omega_j$ and so $\tilde{\omega}(s) = \omega_j(s)$ for all $s \in U_j$. Because

$\|(f \cdot \tilde{\omega})(s) - (f \cdot \omega_j)(s)\| = 0$ for all $s \in U_j$, the vector field $f \cdot \tilde{\omega}$ is within ε of the vector field $f \cdot \omega_j \in \Omega$ on the open set U_j. Thus, by the local uniform approximation property (axiom (IV) in Definition 1.1), $f \cdot \tilde{\omega} \in \Omega$. $\square$

The fact that $\Omega^\perp = \{0\}$ in $\Omega_{\rm wk}$ (Lemma 3.4) suggests that Ω is somehow dense in $\Omega_{\rm wk}$. The next proposition makes this relation more explicit.

Proposition 4.4. *If $\nu \in \Omega_{\rm wk}$ and $\varepsilon > 0$, then there exist a family $\{c_i\}_{i\in\Lambda}$ of pairwise orthogonal projections in $C(\Delta)$ with supremum 1 and a bounded family $\{\omega_i\}_{i\in\Lambda} \subset \Omega$ such that $\|\nu - \sum_{i\in\Lambda} c_i \cdot \omega_i\| < \varepsilon$.*

Proof. By Lemma 2.2, the function $s \mapsto \|\nu(s)\|$ is lower semicontinuous; hence, there exists a meagre set M_ν such that the function $s \mapsto \|\nu(s)\|$ is continuous in the relative topology of $\Delta \setminus M_\nu$. Observe that $\overline{(\Delta \setminus M_\nu)} = \Delta$.

Fix $s_0 \in \Delta \setminus M_\nu$ and let $\omega \in \Omega$ be such that $\omega(s_0) = \nu(s_0)$. Since

$$\|\nu(s) - \omega(s)\|^2 \;=\; \|\nu(s)\|^2 + \|\omega(s)\|^2 - 2{\rm Re}\,\langle \nu, \omega\rangle(s)\,,$$

the continuity in the relative topology of $\Delta \setminus M_\nu$ guarantees the existence of an open subset U_{s_0} of Δ containing s_0 such that $\|\nu(s) - \omega(s)\| < \varepsilon/2$ for all $s \in (\Delta \setminus M_\nu) \cap U_{s_0}$. Hence, again by continuity we get that $\|\nu - \omega\|(s) < \varepsilon$ for all $s \in \overline{U}_{s_0}$. The set $\overline{U}_{s_0}$ is a clopen subset of Δ and $\Delta' = \Delta \setminus \overline{U}_{s_0}$ is also a Stonean space. Further, $M_\nu \cap \Delta' = M_\nu \cap (\Delta \setminus \overline{U}_{s_0})$ is a meagre set such that the function $s \mapsto \|\nu(s)\|$, for $s \in \Delta' \setminus (M_\nu \cap \Delta')$, is continuous in the relative topology.

An application of Zorn's Lemma yields a maximal family $\{(\chi_{U_i}, \omega_i)\}_{i\in\Lambda}$ such that $U_i \cap U_j = \emptyset$ for $i \neq j$ and such that $\|\chi_{U_i}(\nu - \omega_i)\| < \varepsilon$. Maximality ensures that $\overline{(\cup_{i\in I} U_i)} = \Delta$, for otherwise we can enlarge this family by the previous procedure in the Stonean space $\Delta \setminus \overline{(\cup_{i\in\Lambda} U_i)}$. If we let $c_i = \chi_{U_i}$ for $i \in \Lambda$ then it is clear by Lemma 2.2 that $\|\nu - \sum_{i\in\Lambda} c_i \cdot \omega_i\| < \varepsilon$ as for every $j \in \Lambda$ we have that $\|c_j(\nu - \sum_{i\in\Lambda} c_i \cdot \omega_i)\| = \|c_j(\nu - \omega_j)\| < \varepsilon$ and $\bigvee_{i\in\Lambda} c_i = 1$. $\square$

The next result is the key step in the proof of Theorem 4.1.

Proposition 4.5. *For every abelian projection $e \in B(\Omega_{\rm wk})$ and $\varepsilon > 0$ there is an essential ideal $I \subset K(\Omega)$ and $x \in M(I)$ such that $\|e - x\| < \varepsilon$.*

Proof. Assume that $e \in B(\Omega_{\rm wk})$ is an abelian projection and let $\varepsilon > 0$. Thus, by [16, Lemma 13], $e = \Theta_{\nu,\nu}$ for some $\nu \in \Omega_{\rm wk}$ for which $\langle \nu, \nu\rangle$ is a projection of $C(\Delta)$. By Proposition 4.4, there is a family $\{c_i\}_{i\in\Lambda}$ of pairwise orthogonal projections in $C(\Delta)$ with supremum 1 and a bounded family

$\{\omega_j\}_{j\in\Lambda} \subset \Omega$ such that $\|\nu - \tilde{\omega}\| < \varepsilon/(2\|\nu\|)$, where $\tilde{\omega} = \sum_{j\in\Lambda} c_j \cdot \omega_j \in \Omega_{\rm wk}$. Each c_j is the characteristic function of a clopen set U_j and the union U of these sets U_j is dense in Δ.

Let $I = \{a \in K(\Omega) : a(s) = 0, \forall\, s \in \Delta \setminus U\}$, which is an essential ideal of $K(\Omega)$. Define $F^I \subset F_+ \subset K(\Omega)_+$ to be the set

$$F^I = \{\sum_{i=1}^{n} \Theta_{\mu_i,\mu_i} : n \in \mathbb{N},\, \mu_i \in \Omega,\, \mu_i|_{\Delta\setminus U} = 0,\; i = 1,\ldots,n\}.$$

Suppose that $\eta \in \Omega$ satisfies $\|\eta(s)\| = 0$ for all $s \in \Delta \setminus U$, and consider $\Theta_{\eta,\eta} \in F^I$. Observe that $\Theta_{\tilde{\omega},\tilde{\omega}}\,\Theta_{\eta,\eta} = \Theta_{\langle\eta,\tilde{\omega}\rangle\cdot\tilde{\omega},\,\eta}$, which is an element of I because $\langle\eta,\tilde{\omega}\rangle(s) = \langle\eta(s),\tilde{\omega}(s)\rangle = 0$ for all $s \in \Delta\setminus U$ and $\langle\eta,\tilde{\omega}\rangle\cdot\tilde{\omega} \in \Omega$ by Lemma 4.3. Hence, $\Theta_{\tilde{\omega},\tilde{\omega}}$ maps the set F^I back into I. Because F^I is dense in I_+, as we shall show below, $\Theta_{\tilde{\omega},\tilde{\omega}} I \subset I$ and a similar computation shows that $I\Theta_{\tilde{\omega},\tilde{\omega}} \subset I$. Furthermore, writing $x = \Theta_{\tilde{\omega},\tilde{\omega}}$,

$$\|e - x\| = \|\Theta_{\nu,\nu} - \Theta_{\tilde{\omega},\tilde{\omega}}\| \leq (\|\nu\| + \|\tilde{\omega}\|)\,\|\nu - \tilde{\omega}\| < \varepsilon.$$

It remains to show that F^I is dense in I_+. To this end, assume $\varepsilon' > 0$ and $\kappa \in I_+$. Thus, $\kappa(s) = 0$ for all $s \in \Delta\setminus U$. Furthermore, by Lemma 4.2, there exists $h \in F_+$ such that $\|\kappa - h\| < \varepsilon'$. Let $\tilde{h} = \chi_{\Delta\setminus U}\cdot h$ and note that, as $\kappa \in I$, it is also true that $\|\kappa - \tilde{h}\| < \varepsilon'$. Now if h has the form $\sum_{j=1}^{n} \Theta_{\mu_j,\mu_j}$ for some $\mu_j \in \Omega$, then $\tilde{h} = \sum_{j=1}^{n} \Theta_{\chi_{\Delta\setminus U}\mu_j,\chi_{\Delta\setminus U}\mu_j} \in F^I$. $\square$

Proof of Theorem 4.1. Because $K(\Omega)$ is an ideal of A, we have $M(A) \subset M(K(\Omega))$. Moreover, as $K(\Omega)$ is an essential ideal of A we conclude that $M_{\rm loc}(A) = M_{\rm loc}(K(\Omega))$ [2, Proposition 2.3.6]. On the other hand, the inclusions

$$B(\Omega) = M(K(\Omega)) \subset M_{\rm loc}(K(\Omega)) \subset M_{\rm loc}\left(M_{\rm loc}(K(\Omega))\right) \subset B(\Omega_{\rm wk})$$

hold by [11, Theorem 4.6].

Therefore, we are left to show that $M_{\rm loc}\left(M_{\rm loc}(K(\Omega))\right) = B(\Omega_{\rm wk})$. By [11, Corollary 4.3], an element $z \in I(K(\Omega)) = B(\Omega_{\rm wk})$ belongs to $M_{\rm loc}(K(\Omega))$ if and only if for every $\varepsilon > 0$ there is an essential ideal $I \subset K(\Omega)$ and a multiplier $x \in M(I)$ such that $\|z - x\| < \varepsilon$. By Proposition 3.8, $K(\Omega_{\rm wk})$ is the (essential) ideal of $B(\Omega_{\rm wk})$ generated by the abelian projections of $B(\Omega_{\rm wk})$; thus, by Proposition 4.5, $K(\Omega_{\rm wk}) \subset M_{\rm loc}(K(\Omega))$. Hence, $K(\Omega_{\rm wk})$ is an essential ideal of $M_{\rm loc}(K(\Omega))$ and so $M(K(\Omega_{\rm wk})) \subset M_{\rm loc}\left(M_{\rm loc}(K(\Omega))\right)$. However, $B(\Omega_{\rm wk}) = M(K(\Omega_{\rm wk}))$ by Kasparov's Theorem [17, Theorem 2.4] (or by a theorem of Pedersen [20]); hence,

$$B(\Omega_{\rm wk}) = M\left(K(\Omega_{\rm wk})\right) \subset M_{\rm loc}\left(M_{\rm loc}(K(\Omega))\right) \subset B(\Omega_{\rm wk}),$$

which yields $M_{\text{loc}}\left(M_{\text{loc}}(K(\Omega))\right) = B(\Omega_{\text{wk}})$. $\square$

Somerset has shown that every separable postliminal (that is, type I) C*-algebra A has the property that $M_{\text{loc}}(M_{\text{loc}}(A)) = I(A)$ [22, Theorem 2.8]. Theorem 4.1 demonstrates that the same behavior occurs with (certain) nonseparable type I C*-algebras. Somerset's methods are different from ours in at least two ways: he employs the Baire $*$-envelope of a C*-algebra where we use the injective envelope and he uses properties of Polish spaces—spaces that arise from the separability of the algebras under study. It is reasonable to conjecture that $M_{\text{loc}}(M_{\text{loc}}(A)) = I(A)$ for all C*-algebras A that possess a postliminal essential ideal. To prove such a statement, it would be enough to prove it for any continuous trace C*-algebra A.

5 Direct Sum Decompositions

A Kaplansky–Hilbert module E over $C(\Delta)$ is said to be *homogeneous* [16] if there is a subset $\{\nu_j\}_{j\in\Lambda} \subset E$ – called an *orthonormal basis* – such that $\langle \nu_i, \nu_j\rangle = 0$ for all $j \neq i$, $|\nu_j| = 1$ for all j, and $\{\nu_j\}_{j\in\Lambda}^{\perp} = \{0\}$, where for any $\nu \in E$, $|\nu|$ is the continuous real-valued function $|\nu| = \langle \nu, \nu\rangle^{1/2} \in C(\Delta)$.

Kaplansky introduced the notion of homogeneous AW*-module with the aim of reducing the study of abstract AW*-modules to the slightly more concrete setting in which the modules have an orthonormal basis. This is justified by the following result:

Theorem 5.1 ([16]). *Let E be a Kaplansky-Hilbert module over $C(\Delta)$. Then there exist orthogonal projections $\{c_i\}_{i\in I} \subset C(\Delta)$ with supremum 1 such that $c_i\, E$ is a homogenous AW*-module over $c_i\, C(\Delta)$.*

Note that in the situation of Theorem 5.1, for each i there exists a clopen set $\Delta_i \subset \Delta$ with $c_i = \chi_{\Delta_i}$. The sets $\{\Delta_i\}$ are pairwise disjoint, and $\cup_i \Delta_i$ is dense in Δ.

In this section we consider the effect of a direct sum decomposition in the structures that have been studied in the previous sections, namely the Fell algebra A of the weakly continuous Hilbert bundle $(\Delta, \{H_s\}_{s\in\Delta}, \Omega, \Omega_{\text{wk}})$, and its local multiplier algebra $M_{\text{loc}}(A)$. We show that a decomposition of Ω_{wk} into a direct sum $\oplus_i c_i \Omega_{\text{wk}}$ given by a partition of the identity $\{c_i\}$ in $C(\Delta)$ leads one to consider two corresponding direct sum C*-algebras: $\oplus_i A_i$ and $\oplus_i M_{\text{loc}}(A_i)$, where A_i is a subalgebra of A for all i. We prove that A need not be isomorphic to $\oplus_i A_i$, yet $M_{\text{loc}}(A) \cong \oplus_i M_{\text{loc}}(A_i)$. The latter result is especially interesting if one recalls that $M_{\text{loc}}(A)$ is generally not an AW*-algebra [3, Theorem 6.13].

Theorem 5.2. *Let $(\Delta, \{H_s\}_{s\in\Delta}, \Omega)$ be a continuous Hilbert bundle over the Stonean space Δ. Assume that $\{\Delta_i\}_{i\in I}$ is a family of pairwise-disjoint clopen subsets of Δ whose union is dense in Δ, and for each $i \in I$ let $c_i = \chi_{\Delta_i} \in C(\Delta)$ and $\Omega_i = \{\omega_{|\Delta_i} : \omega \in \Omega\}$. Then:*

i. *$(\Delta_i, \{H_s\}_{s\in\Delta_i}, \Omega_i)$ is a continuous Hilbert bundle;*

ii. *$(\Omega_i)_{\mathrm{wk}} \cong c_i \cdot \Omega_{\mathrm{wk}}$ as C^*-modules;*

iii. *$\Omega_{\mathrm{wk}} \cong \bigoplus_i (\Omega_i)_{\mathrm{wk}}$ as C^*-modules;*

iv. *$B((\Omega_i)_{\mathrm{wk}}) \cong c_i \cdot B(\Omega_{\mathrm{wk}})$ as C^*-algebras;*

v. *$B(\Omega_{\mathrm{wk}}) \cong \bigoplus_i B((\Omega_i)_{\mathrm{wk}})$ as C^*-algebras.*

*In **ii** and **iii**, the isomorphism is considered together with the identification $C(\Delta_i) \simeq c_i\, C(\Delta)$.*

Proof. Being clopen in Δ, each Δ_i is itself a Stonean space, and it is easy to see that $C(\Delta_i) \cong c_i\, C(\Delta)$
i. For axiom (I) in Definition 1.1, we aim to show that Ω_i is a $C(\Delta_i)$ module. Let $\omega \in \Omega$ and consider $\omega_i = \omega|_{\Delta_i}$. Choose any $f_i \in C(\Delta_i)$. As Δ_i is clopen, f_i can be extended to $F_i \in C(\Delta)$ such that $f_i = F_i|_{\Delta_i}$, and $F_i|_{\Delta\setminus\Delta_i} = 0$. The action $f_i \cdot \omega_i = (F_i \cdot \omega)|_{\Delta_i}$ gives Ω_i the structure of a $C(\Delta_i)$ module. Axioms (II) and (III) of Definition 1.1 are trivially satisfied.

For axiom (IV), let $\xi : \Delta_i \to \bigsqcup_{s\in\Delta_i} H_s$ be a vector field such that for every $s_0 \in \Delta_i$ and $\varepsilon > 0$ there is an open set $U_i \subset \Delta_i$ containing s_0 and a $\omega_i \in \Omega_i$ with $\|\omega_i(s) - \xi(s)\| < \varepsilon$ for all $s \in U_i$. Let $\Xi : \Delta_i \to \bigsqcup_{s\in\Delta} H_s$ be the vector field that coincides with ξ on Δ_i and is identically zero off Δ_i. By the definition of Ω_i, there is $\omega \in \Omega$ such that $\omega_i = \omega|_{\Delta_i}$. The set U_i is also open in Δ, and $\|\omega(s) - \Xi(s)\| < \varepsilon$ for all $s \in U_i$. If $s_0 \notin \Delta_i$ choose any open set V_i containing s_0 such that $V_i \cap U_i = \emptyset$ and let $\omega \in \Omega$ be arbitrary; then $0 = \|\chi_{\Delta_i}(s)\omega(s) - \Xi(s)\| < \varepsilon$ for all $s \in V_i$. Since $\chi_{\Delta_i} \cdot \omega \in \Omega$ and since Ω is closed under local uniform approximation, $\Xi \in \Omega$, whence $\xi \in \Omega_i$.
ii. Let $T_i : c_i\, \Omega_{\mathrm{wk}} \to (\Omega_i)_{\mathrm{wk}}$ be given by $T_i(c_i\nu) = \nu|_{\Delta_i}$. It is clear that T_i is well defined, linear, bounded, and has trivial kernel; to show that it is onto, note that if $\nu_i \in (\Omega_i)_{\mathrm{wk}}$, then—since Δ_i is clopen—the vector field $\nu : \Delta \to \bigsqcup_{s\in\Delta} H_s$ defined by $\nu(s) = 0$, for $s \notin \Delta_i$, and $\nu(s) = \nu_i(s)$, for $s \in \Delta_i$, has the property that $\langle \omega, \nu \rangle \in C(\Delta)$, for all $\omega \in \Omega$; so $\nu \in \Omega_{\mathrm{wk}}$ and $\nu_i = T_i(c_i\nu)$. It is also easy to check that T_i preserves inner products.
iii. Let $T : \Omega_{\mathrm{wk}} \to \bigoplus_i (\Omega_i)_{\mathrm{wk}}$, given by $T\nu = (T_i(c_i\nu))_{i\in I}$. The previous paragraph and Lemma 2.1 show that T is an isometry; we show now that T

is onto. Suppose that $\nu' = (\nu_i)_{i\in I} \in \bigoplus_i (\Omega_i)_{\rm wk}$. For each $i \in I$ let $\tilde{\nu}_i$ denote the vector field on Δ that coincides with ν_i on Δ_i and vanishes elsewhere. Then $\tilde{\nu}_i \in \Omega_{\rm wk}$ and $T_i(c_i\tilde{\nu}_i) = \nu_i$. Hence, if $\nu = \sum_i c_i\tilde{\nu}_i$ as in Remark 2.5, we have $\nu \in \Omega_{\rm wk}$ and $T\nu = \nu'$. Thus, $\Omega_{\rm wk}$ and $\bigoplus_i (\Omega_i)_{\rm wk}$ are isomorphic Banach spaces. Similar arguments show that $\bigoplus_i (\Omega_i)_{\rm wk}$ is a $C(\Delta)$-module and that T is module isomorphism. Hence, $\Omega_{\rm wk} \cong \bigoplus_i (\Omega_i)_{\rm wk}$ as C*-modules.
iv. Let $\rho_i : c_i\, B(\Omega_{\rm wk}) \to B((\Omega_i)_{\rm wk})$ be given by $\rho_i(c_i b)\, T_i(c_i\nu) = (b\nu)|_{\Delta_i}$. This map is well-defined because if $c_ib_1 = c_ib_2$ then for any $\nu \in \Omega_{\rm wk}$ we have $(b_1\nu)|_{\Delta_i} = (c_ib_1\nu)|_{\Delta_i} = (c_ib_2\nu)|_{\Delta_i} = (b_2\nu)|_{\Delta_i}$. A similar computation shows that ρ_i is one-to-one, and linearity is clear. To see that ρ_i is onto, let $b_i \in B((\Omega_i)_{\rm wk})$. Consider the injection $\tilde{}: (\Omega_i)_{\rm wk} \to \Omega_{\rm wk}$ where $\tilde{\nu_i} \in \Omega_{\rm wk}$ is the vector field that agrees with ν_i on Δ_i and is 0 elsewhere. Let $b \in B(\Omega_{\rm wk})$ be the operator given by $b\nu = \widetilde{b_i(\nu|_{\Delta_i})}$. Then $\rho_i(c_ib)(T_ic_i\nu) = (b\nu)|_{\Delta_i} = \widetilde{b_i(\nu|_{\Delta_i})}|_{\Delta_i} = b_i(\nu|_{\Delta_i}) = b_i\,(T_ic_i\nu)$, so $\rho_i(c_ib) = b_i$.
v. Let $\rho : B(\Omega_{\rm wk}) \to \bigoplus_i B((\Omega_i)_{\rm wk})$ be the map $\rho(b) = (\rho_i(c_ib))_{i\in I}$. It is clear that ρ is a homomorphism. If $\rho(b) = 0$ for some $b \in B(\Omega_{\rm wk})$, then – as each ρ_i is one-to-one – $c_ib = 0$ for all i; this implies that $b^*b = b^*(\sup_i(c_i \cdot I))b = \sup_i(b^*c_ib) = 0$ by [14, Corollary 4.10], so $b = 0$ and ρ is one-to-one. To show that ρ is onto, let $(b_i)_i \in \bigoplus_i B((\Omega_i)_{\rm wk})$; as each ρ_i is onto, there exist operators $b^i \in B(\Omega_{\rm wk})$ with $\rho_i(c_ib^i) = b_i$. Define $b \in B(\Omega_{\rm wk})$ by $b\nu = \sum_i c_ib^i\nu$ (in the sense of Remark 2.5; that is, $c_ib\nu = c_ib^i\nu$). Then $\rho_i(c_ib)\nu|_{\Delta_i} = (c_ib\nu)|_{\Delta_i} = (c_ib^i\nu)|_{\Delta_i} = \rho_i(c_ib^i)\nu|_{\Delta_i} = b_i\nu|_{\Delta_i}$. So $\rho(b) = (b_i)_i$. ☐

Proposition 5.3. *Assume the notation, hypotheses, and conclusions of Theorem 5.2. Then there exists an example where the canonical embedding $\Omega \hookrightarrow \bigoplus_i \Omega_i$ (via the isometry T from the proof of **iii** in Theorem 5.2) is not onto. In particular, Ω is properly contained in $\Omega_{\rm wk}$.*

Proof. Take Δ and the family of clopen subsets $\{\Delta_i\}_{i\in I}$ in Theorem 5.2 to be such that $\bigcup_{i\in I} \Delta_i \neq \Delta$. Thus, I is an infinite set. Let H be a Hilbert space with orthonormal basis $\{e_i\}_{i\in I}$ and consider the trivial Hilbert bundle $\Omega = C(\Delta, H)$ of all continuous functions $\omega : \Delta \to H$. As in Theorem 5.2, let $\Omega_i = C(\Delta_i, H)$.

For each $i \in I$, set $\omega_i \in \Omega$ with $\omega_i(s) = e_i$ for all s and consider $(\omega_i)_{i\in I} \in \bigoplus_i \Omega_i$. Under the isomorphism of Theorem 5.2, this element $(\omega_i)_{i\in I}$ is identified with $\omega = \sum_{i\in I} \chi_{\Delta_i} \cdot \tilde{\omega}_i \in \Omega_{\rm wk}$ (in the sense of Remark 2.5), where $\tilde{\omega}_i$ is any element of Ω that agrees with ω_i on Δ_i and vanishes off Δ_i. Under this identification, $\omega \notin \Omega$; that is, the function $s \mapsto \|\omega(s)\|$ fails to be continuous on Δ. We argue this by contradiction.

Assume that $s \mapsto \|\omega(s)\|$ is continuous on Δ. Because $\|\omega(s)\| = 1$ for all $s \in \cup_{i\in I}\Delta_i$, continuity implies that $\|\omega(s)\| = 1$ for $s \in \Delta$. Choose $s_0 \in \Delta \setminus (\cup_{i\in I}\Delta_i)$ and let $(s_\alpha)_{\alpha\in\Lambda} \subset \cup_{i\in I}\Delta_i$ be a net such that $s_\alpha \to s_0$. Let $\eta \in \Omega$ be the constant field $\eta(s) = \omega(s_0)$, for all $s \in \Delta$. Since $\omega \in \Omega_{\mathrm{wk}}$, we have

$$\lim_{\alpha} \langle \omega(s_\alpha), \eta(s_\alpha)\rangle \;=\; \langle \omega(s_0), \eta(s_0)\rangle \;=\; \langle \omega(s_0), \omega(s_0)\rangle \;=\; 1\,. \tag{5}$$

For each $\alpha \in \Lambda$ let $i(\alpha) \in I$ be such that $s_\alpha \in \Delta_{i(\alpha)}$. Thus, for every $\alpha \in \Lambda$, $I_\alpha = \{i(\beta) : \; \beta \in I, \; \beta \geq \alpha\}$ is an infinite set (for otherwise $s_0 \in \Delta_i$ for some $i \in I$). Therefore,

$$\lim_{\alpha} \langle \omega(s_\alpha), \eta(s_\alpha)\rangle \;=\; \lim_{\alpha} \langle e_{i(\alpha)}, \omega(s_0)\rangle \;=\; 0\,. \tag{6}$$

As (5) and (6) cannot be true simultaneously, we obtain a contradiction. Hence, $\omega \notin \Omega$. ☐

Our second reduction theorem below notes some consequences of Theorem 5.2 when applied to the injective envelope and local multiplier algebras of the Fell algebra A associated to a continuous Hilbert bundle.

Theorem 5.4. *Let $(\Delta, \{H_t\}_{t\in\Delta}, \Omega)$ be a continuous Hilbert bundle over the Stonean space Δ and let $A = (\Delta, \{K(H_t\}, \Gamma)$ denote the associated continuous trace C*-algebra of Fell. Assume that $\{\Delta_i\}_{i\in I}$ is a family of pairwise-disjoint clopen subsets of Δ whose union is dense in Δ, and for each $i \in I$ let $c_i = \chi_{\Delta_i} \in C(\Delta)$ and $\Omega_i = \{\omega_{|\Delta_i} : \omega \in \Omega\}$. Then:*

i. *if A_i denotes the Fell algebra of $(\Delta_i, \{H_s\}_{s\in\Delta_i}, \Omega_i)$, then $A_i \cong c_i \cdot A$;*

ii. *$I(A_i) = B((\Omega_i)_{\mathrm{wk}})$;*

iii. *$I(A) \cong \bigoplus_{i\in I} I(A_i)$;*

iv. *$M_{\mathrm{loc}}(A) \cong \bigoplus_{i\in I} M_{\mathrm{loc}}(A_i)$.*

Proof. Let $A_i = (\Delta_i, \{K(H_s)\}_{s\in\Delta}, \Gamma_i)$ denote the Fell C*-algebra associated to the Hilbert bundle $(\Delta_i, \{H_s\}_{s\in\Delta_i}, \Omega_i)$. That is, Γ_i consists of all weakly continuous almost finite-dimensional operator fields $a_i : \Delta_i \to \bigsqcup_{s\in\Delta_i} K(H_s)$ such that $s \mapsto \|a_i(s)\|$ is continuous. We have that $B((\Omega_i)_{\mathrm{wk}})$ is a type I AW*-algebra with centre $C(\Delta_i)$.

i. For each $a_i \in \Gamma_i$ there is an $a \in \Gamma$ such that $a_i = a|_{\Delta_i}$. To verify this, let $a : \Delta_i \to \bigsqcup_{s\in\Delta} K(H_s)$ be the operator field defined by $a(s) = a_i(s)$, for $s \in \Delta_i$, and $a(s) = 0$, for $s \notin \Delta_i$. Since Δ_i is a clopen set, the maps

$s \to \|a(s)\|$ and $s \mapsto \langle a(s)\omega_1(s), \omega_2(s)\rangle$ are continuous for every $\omega_1, \omega_2 \in \Omega$. The operator field a is also locally finite-dimensional, again because Δ_i is clopen and a_i has the property on Δ_i. Hence, $a \in \Gamma$. Next, let $\pi_i : A_i \to c_i A$ be defined by $\pi_i(a_i) = c_i a$, where $a \in A$ is any operator field that restricts to a_i on Δ_i. This map is clearly well-defined, and a homomorphism.
ii. By Theorem 3.1, $B((\Omega_i)_{\mathrm{wk}}) = I(A_i) = I(c_i A)$.
iii. By [14, Lemma 6.2], $I(c_i A) = c_i I(A)$. Hence, $I(A_i) = B((\Omega_i)_{\mathrm{wk}})$ and Theorem 5.2 immediately yields $I(A) \cong \bigoplus_{i\in I} I(A_i)$.
iv. We take each $M_{\mathrm{loc}}(A_i)$ to be a C*-subalgebra of $B((\Omega_i)_{\mathrm{wk}})$. First we remark that the isomorphism ρ from Theorem 5.2 sends A into $\bigoplus_i A_i$. To see why, recall that $a\nu(s) = a(s)\nu(s)$, for all $a \in A$, $\nu \in \Omega_{\mathrm{wk}}$, and $s \in \Delta$ (Proposition 3.6). Since, for a given $i \in I$, the action of $\rho_i(a)$ on $\nu_i \in (\Omega_i)_{\mathrm{wk}}$ is defined by $\nu_i \mapsto (a\nu)|_{\Delta_i}$, where $\nu \in \Omega_{\mathrm{wk}}$ is any vector with $\nu|_{\Delta_i} = \nu_i$, it is easy to verify that $\rho_i(a)$ is a weakly continuous almost finite-dimensional operator field on Δ_i.

To show that $\rho\left(M_{\mathrm{loc}}(A)\right) \subset \bigoplus_i M_{\mathrm{loc}}(A_i)$, let $x \in M_{\mathrm{loc}}(A) \subset I(A)$ and suppose that $\varepsilon > 0$. Thus, there is an essential ideal $J \subset A$ and a multiplier $x \in M(J)$ such that $\|x - y\| < \varepsilon$. Further, there exists an open dense subset $U \subset \Delta$ such that

$$J \,=\, \{a \in A : a(s) = 0\,,\; s \in \Delta \setminus U\}\,. \tag{7}$$

For $i \in I$, let $U_i = \Delta_i \cap U$, which is an open dense set in Δ_i. Therefore,

$$J_i \,=\, \{a_i \in A_i :\, a(s) = 0,\; s \in \Delta_i \setminus U_i\} \tag{8}$$

is an essential ideal in A_i. We aim to show that $\rho_i(y) \in M(J_i)$. To this end, select $a_i \in J_i$. As $A_i \cong c_i \cdot A$, there is an $a \in A$ such that $a_i(s) = a(s)$ for all $s \in \Delta_i$. Moreover, $a \in A$ can be chosen so that $a(s) = 0$ for all $s \in \Delta \setminus \Delta_i$.

Because $a_i \in J_i$, we conclude that $a(s) = 0$ for all $s \in \Delta \setminus U$; that is, $a \in J$. Therefore, $ya \in J$, which implies that $ya(s) = 0$ for all $s \in \Delta \setminus U$. In particular, $ya(s) = 0$ for all $s \in \Delta_i \setminus U_i$. The element $\rho_i(y)a_i \in B((\Omega_i)_{\mathrm{wk}})$ is in fact an operator field since $\rho_i(y)a_i = \rho_i(y)\rho_i(c_i a) = \rho_i(c_i(ya)) \in A_i$. Then, for all $s \in \Delta_i \setminus U_i$ and $\nu \in \Omega_{\mathrm{wk}}$,

$$\begin{aligned}
[\rho_i(y)a_i](s)(T_i c_i \nu)(s) &= \rho_i(y)a_i(T_i c_i \nu)(s) = \rho_i(c_i y a)(T_i c_i \nu)(s) \\
&= (ya)\nu|_{\Delta_i}(s) = (ya)(s)\nu|_{\Delta_i}(s) = 0.
\end{aligned}$$

With ν being arbitrary, we conclude that $\rho_i(y)a_i(s) = 0$, that is $\rho_i(y)a_i \in J_i$, and so $\rho_i(y)$ is a left multiplier of J_i. By a similar argument, $\rho_i(y)$ is a right multiplier of J_i, and so $\rho_i(y) \in M(J_i)$. Thus, $\rho(y) \in \bigoplus_i M_{\mathrm{loc}}(A_i)$ and

$\|\rho(x) - \rho(y)\| = \|x - y\| < \varepsilon$. As $\varepsilon > 0$ was chosen arbitrarily, this proves that $\rho(x) \in \bigoplus_i M_{\mathrm{loc}}(A_i)$.

Conversely, let us show that $\bigoplus_i M_{\mathrm{loc}}(A_i) \subset \rho(M_{\mathrm{loc}}(A))$. Let $(x_i)_i \in \bigoplus_i M_{\mathrm{loc}}(A_i)$; thus for each $i \in I$, there exist an essential ideal $J_i \subset A_i$ and $y_i \in M(J_i)$ such that $\|x_i - y_i\| < \varepsilon$ for all $i \in I$. For each $i \in I$, there exists an open dense subset $U_i \subset \Delta_i$ such that J_i is given as in (8). Define $U = \bigcup_{i \in I} U_i$, which is an open dense subset of Δ and let J be the essential ideal of A defined as in (7) (for our present choice of U). Let $y \in B(\Omega_{\mathrm{wk}})$ be such that $\rho(y) = (y_i)_i$.

For each $\omega \in \Omega$, we have that $y\omega \in \Omega_{\mathrm{wk}}$.

Claim 1. If $\omega \in \Omega$ is such that $\omega(s) = 0$ for all $s \in \Delta \setminus U$, then $y\omega \in \Omega$ and $y\omega(s) = 0$ for $s \in \Delta \setminus U$.

Assuming Claim 1, consider the set $F_+ = \mathrm{span}\,\{\Theta_{\omega,\omega} : \ \omega \in \Omega, \ \omega(s) = 0 \ \text{ for } \ s \in \Delta \setminus U\}$, which by Lemma 4.2 is dense in K_+, where K is the essential ideal of $K(\Omega)$ defined by $K = K(\Omega) \cap J$. By the Claim, $y\Theta_{\omega,\omega} = \Theta_{y\omega,\omega} \in K$ for all $\omega \in \Omega$. Therefore, y is a left multiplier of K. Similarly, y is a right multiplier of K , which yields $y \in M(K)$. Hence, $(x_i)_{i\in I}$ is within ε of a multiplier—namely, $\rho(y)$—of an essential ideal of $\rho\,(K(\Omega))$. Thus, by the Frank–Paulsen description of local multiplier algebras [11], $(x_i)_{i\in I} \in \rho\,(M_{\mathrm{loc}}(K(\Omega)))$. By Theorem 4.1, $M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(K(\Omega))$, so $(x_i)_{i\in I} \in \rho(M_{\mathrm{loc}}(A))$.

We are now left with proving Claim 1. Assume that $\omega \in \Omega$ with $\omega(s) = 0$ for all $s \in \Delta \setminus U$. Let $i \in I$ and let $\omega_i = \omega|_{\Delta_i} \in \Omega_i$. Note that for every $\eta_i \in \Omega_i$, $\Theta_{\omega_i,\eta_i} \in J_i$, and hence $\Theta_{y_i\omega_i,\eta_i} = y_i\Theta_{\omega_i,\eta_i} \in J_i$. Also, $y_i\omega_i \in \Omega_i$. Indeed, suppose that $s_0 \in \Delta_i$ and let $\eta_i \in \Omega_i$ such that $\|\eta_i(s_0)\| = 1$. Choose a clopen subset $V_i \subset \Delta_i$ of s_0 for which $\|\eta_i(s)\| \geq 1/2$ for all $s \in V_i$ and define $f(s) = \chi_{V_i}(s)\|\eta_i(s)\|^{-2}$. Thus, $f \in C(\Delta_i)$ and so $f \cdot \eta_i \in \Omega_i$. Then, since $\Theta_{y_i\omega_i,\eta_i} \in J_i \subset A_i$, we have $\Theta_{y_i\omega_i,\eta_i}(f \cdot \eta_i) \in \Omega_i$. So $\chi_{V_i} \cdot y_i\omega_i = \Theta_{y_i\omega_i,\eta_i}(f \cdot \eta_i) \in \Omega_i$. Thus, $y_i\,\omega_i$ is a local uniform limit of vectors fields in Ω_i and hence, $y_i\,\omega_i \in \Omega_i$. Moreover, since $\Theta_{y_i\omega_i,\eta_i} \in J_i$ for any $\eta_i \in \Omega_i$, we have $y_i\omega_i(s) = 0$ for $s \in \Delta_i \setminus U_i$.

Since $(y\omega)(s) = (y_i\,\omega_i)(s)$ for $s \in \Delta_i$, the lower semicontinuous function $s \mapsto \|(y\omega)(s)\|$ is continuous on $\bigcup_i \Delta_i$ and vanishes on $(\bigcup_i \Delta_i) \setminus U$.

Claim 2. There exists $C > 0$ such that $\|y\omega(s)\| \leq C\,\|\omega(s)\|$, $s \in \Delta_i$, $i \in I$.

We will use Claim 2 to show that the function $s \mapsto \|(y\omega)(s)\|$ is continuous on Δ. Let $s \in \Delta \setminus (\bigcup_i \Delta_i)$ and let $(s_\alpha)_\alpha \subset \bigcup_i \Delta_i$ be a net such that $s_\alpha \to s$ in Δ. This implies that $\lim_\alpha \|\omega(s_\alpha)\| = 0$. By lower semicontinuity

of the function $s \mapsto \|(y\omega)(s)\|$,

$$0 \le \|y\omega(s)\| \le \lim_{\alpha} \|y\omega(s_\alpha)\| \le C \ \lim_{\alpha} \|\omega(s_\alpha)\| = 0\,,$$

and it follows that $s \mapsto \|(y\omega)(s)\|$ is continuous on Δ and vanishes in $\Delta \setminus U$. This establishes Claim 1.

We finish the proof by proving Claim 2. Fix $s \in \Delta_i$, and let $C = \sup_i \|y_i\|$. We already know that $y_i\omega_i \in \Omega_i$, and so

$$\begin{aligned}\|y\omega(s)\| = \|y_i\omega_i(s)\| = \|y_i\omega_i\|(s) \le \|y_i\|\,\|\omega_i\|(s) \\ \le C\,\|\omega_i\|(s) = C\,\|\omega_i(s)\| = C\,\|\omega(s)\|.\square\end{aligned}$$

Local multiplier algebras behave well under direct sums: $M_{\mathrm{loc}}(\oplus_i A_i) \cong \oplus_i M_{\mathrm{loc}}(A_i)$ [2, Proposition 2.3.6]. However, the isomorphism of local multiplier algebras in Theorem 5.4 cannot be established via that generic result:

Proposition 5.5. *Assume the notation, hypotheses, and conclusions of Theorem 5.4. Although ρ sends A into $\bigoplus_i A_i$, it need not be true that $A \cong \bigoplus_i A_i$.*

Proof. If Δ and Ω are as in Proposition 5.3, then $\rho(\Theta_{\omega,\omega}) = (\Theta_{\omega_i,\omega_i})_{i\in I} \in \oplus_{i\in I} A_i$, but $\rho(\Theta_{\omega,\omega}) \notin \rho(A)$. $\square$

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